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Some Remarks on Nonequilibrium Dynamics of Infinite Particle Systems

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Classical mechanics of infinitely many particles in dimensions one and two is considered, particles interacting by a superstable pair potential of finite range. The group of motion generated by Newton's equations is constructed in the space of locally finite configurations with a logarithmic order of energy fluctuations at infinity. A core of the Liouville operator is also described. Results of Dobrushin and the author and of Marchioro–Pellegrinotti–Pulvirenti are improved.

KEY WORDS: Infinite systems; superstable interactions; Liouville operator; essential self-adjointness.

1. INTRODUCTION

In the last 15 years several papers have been devoted to nonequilibrium dynamics of infinite particle systems. Nevertheless, the most fundamental problems including the existence of three-dimensional dynamics are still unsolved; known results are far from being perfect even from an aesthetical point of view. For instance, two-dimensional dynamics has been constructed only for two particular classes of interactions,[3] and relation of the dynamics to its formal generator, i.e., to the Liouville operator L, is clear only in case of one-dimensional particles interacting with a potential with hard core; see Ref. 10. The main purpose of this paper is to remove restrictive conditions of this kind. We are going to prove the existence of

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the dynamics for superstable interactions of finite range in dimensions one and two; cf. Refs. 2, 3, and 6. The very same method applies in case of anharmonic systems; cf. Refs. 9, 12, and 13. Basic ideas are essentially the same as those of Refs. 2 and 3; certain technical tricks developed in Refs. 4 and 5 are used to control boundary effects. The transition group T_t of temporal evolution will be constructed in an explicitly defined set Ω of allowed configurations characterized by a logarithmic order of energy fluctuations. Furthermore, a dense class $\mathbb{C}_1(\Omega)$ of quasilocal functions $\Omega \to \mathbb{R}$ will be defined such that the Liouville operator is well defined on $\mathbb{C}_1(\Omega)$ and $T_t\mathbb{C}_1(\Omega) = \mathbb{C}_1(\Omega)$, whence essential self-adjointness of iL and some further useful properties of the dynamics follow directly.

We are going to investigate the motion of a countable collection I of identical particles of unit mass in the d-dimensional Euclidean space $\mathbb{R}^d$ with $d = 1, 2$. The usual norm and inner product of $\mathbb{R}^d$ will be denoted by $|\cdot|$ and $\langle \cdot, \cdot \rangle$, respectively. Configurations of the system are represented as infinite sequences $\omega = (q_k, p_k)_{k \in I}$, where $q_k \in \mathbb{R}^d$ and $p_k \in \mathbb{R}^d$ are the position and velocity of the particle labeled by $k \in I$, i.e., $\omega \in (\mathbb{R}^{2d})^I$. If necessary then the more informative notation $q_k = q_k(\omega)$, $p_k = p_k(\omega)$ will be used. Only locally finite configurations are allowed, i.e., the sequence $(q_k)_{k \in I}$ of positions may not have limit points at all, but some additional restrictions are necessary, too. We assume that our particles interact by a symmetric pair potential $U: \mathbb{R}^d \to (-\infty, +\infty]$ of radius $r > 0$ of interaction, i.e., $U(x) = U(-x)$ and $U(x) = 0$ if $|x| > r$. Let $\operatorname{grad} U$ denote the vector of partial derivatives of U; then the equations of motion read as

$$\frac{dp_k}{dt} = -\sum_{j \neq k} \operatorname{grad} U(q_k - q_j), \qquad \frac{dq_k}{dt} = p_k, \qquad k \in I \tag{1.1}$$

The following regularity properties of the interaction potential are needed in the proof of existence of the transition group of motion. The potential may have a singularity at 0; then $U(0) = +\infty$ and $\lim U(x) = +\infty$ as $|x| \to +\infty$, but U is continuously differentiable for $x \neq 0$. If U is bounded then continuous differentiability of U is assumed for all $x \in \mathbb{R}^d$. Finiteness of the range of U can certainly be weakened, but the present proof seems to be optimal in the case of interactions of finite range. It is more important that singularity of U, if any, cannot be too strong. Indeed, in a dense medium the velocity of propagation of shock waves depends on the strength of the interaction, and this velocity becomes arbitrarily large in the extreme case of hard cores. The related very intensive transfer of energy from infinity towards the center may result in an explosion of the system in a finite time. One-dimensional geometry does not allow such a critical accumulation of energy, but explosion of infinite systems of hard spheres on the plane can be demonstrated by means of simple examples. Thus, if

$d = 2$ then interactions with a hard core are excluded, we assume that

$$|x|\,|\operatorname{grad} U(x)| \leqslant a + bU(x) \tag{1.2}$$

holds with some positive constants a and b. From a mathematical point of view (1.2) means that the singularity of U cannot be stronger than that of $|x|^{-b}$. We shall see later that under (1.2) the velocity of energy transfer is of the order of the square root of energy on the spot, which is essentially the same as velocity of energy transport due to an ordered flow of particles. Just as in equilibrium theory,[16,17] we need superstability of the interaction to control the number of particles. Let $A \geqslant 0$, $B > 0$ and suppose that

$$\sum_{k=1}^{n} \sum_{j \neq k} U(q_k - q_j) \geqslant -An + BN \tag{1.3}$$

holds for any finite collection $q_1, q_2, \ldots, q_n$ of points of $\mathbb{R}^d$, where N denotes the number of pairs $[k, j]$ such that $|q_k - q_j| \leqslant r$. Notice that (1.3) is equivalent to condition (SS) of Ref. 17, but it seems to be a little bit stronger than superstability as defined in Ref. 16. Of course, both conditions can be verified under some natural assumptions, e.g., if U is not integrable near 0; cf. Refs. 16 and 17 with some further references. Let us remark that for pair potentials of finite range (1.3) is a general sufficient condition for the existence of Gibbs random fields with interaction U; see Ref. 17. Finally, the following local Lipschitz condition will be used in the study of contraction properties of the right-hand side of (1.1); it is convenient to control Lipschitz continuity of (1.1) in terms of potential energy. Let $L > 1 + 2a/b$ and suppose that

$$|\operatorname{grad} U(x) - \operatorname{grad} U(y)| \leqslant L\left[L + U(x) + U(y)\right]^c |x - y| \tag{1.4}$$

holds for all $x, y \neq 0$ with some constant $c \geqslant 0$. Notice that $L + U(x) + U(y) > 0$ in view of (1.2). If U is bounded then $c = 0$ may be assumed, but $c > 1$ is necessary in the singular case. (1.4) means that the singularity of U, if any, cannot be too weak, logarithmic singularities are excluded. For a more general form of (1.4) see (U) in Ref. 3. The validity of (1.2), (1.3), (1.4) will be supposed throughout this paper.

As is usual in the theory of infinite systems, solutions to (1.1) are constructed as limits of solutions to finite subsystems of (1.1). The passage to the infinite system is based on an *a priori* bound expressing a local version of the law of energy conservation. Heuristic ideas behind the *a priori* bound are the following (cf. Ref. 2). Consider the total energy H of particles in a box V; then H is proportional to the volume of V, while the energy of external particles interacting with internal ones is proportional to the surface of V, i.e., to $H^{1-1/d}$. In view of the law of energy conservation, only boundary effects influence the value of H, namely, the work of external forces and the transport of energy through the surface of V. On the other hand, (1.2) implies that velocities of both kinds of energy flow are of the same order, $H^{1/2}$, thus in the least favorable case the order of dH/dt may be as large as $H^{1/2}H^{1-1/d} = H^{3/2-1/d}$. Since $du/dt = u^{\lambda}$ has global solutions only if $\lambda \leqslant 1$, we can hope for an *a priori* bound in terms of total energy only in dimensions one and two. A nonrigorous example outlined in Ref. 3 suggests that such a bound may not exist in the three-dimensional case, because critical accumulation of energy may result in an explosion of the system in a finite time. A mathematical realization of the ideas above is not quite trivial. Just as in Refs. 2 and 3, energy flow will be controlled by means of a partial differential inequality formulated in terms of a spatial cutoff of total energy. The new trick of the present proof of the *a priori* bound is a more effective—and more sophisticated—definition of the spatial cutoff; cf. Refs. 4 and 5.

The construction of the core of the Liouville operator is based on an analysis of dependence of solutions on initial data. This approach is fairly general and applies to any deterministic model satisfying a reasonable *a priori* bound. Since even the local space-time behavior of solutions depends on energy level of the initial configuration, a scale of uniform norms of quasilocal functions corresponding to increasing energy levels seems to be the right tool in the study of the transition group; see Ref. 4. The problem, however, is more complex in case of interacting diffusion processes. Related questions are to be discussed elsewhere.

2. MAIN RESULT

The space Ω of allowed configurations is defined in the following way. Let $g(u) = 1 + \log(1+u)$ and let $H(\omega, \mu, \sigma)$ denote total energy plus a multiple of particle number in the sphere with center $\mu \in \mathbb{R}^d$ and radius $\sigma > 0$, i.e.,

$$H(\omega, \mu, \sigma) = \frac{1}{2} \sum_{|q_k - \mu| \leqslant \sigma} \left[|p_k|^2 + A + \sum_{\substack{j \neq k \\ |q_j - \mu| \leqslant \sigma}} U(q_k - q_j) \right] \tag{2.1}$$

with $A \geqslant 0$ being the same as in (1.3), then

$$\overline{H}(\omega) = \sup_{\mu} \sup_{\sigma \geqslant g(|\mu|)} \sigma^{-d} H(\omega, \mu, \sigma) \tag{2.2}$$

is called the logarithmic fluctuation of energy, or energy level of ω, and Ω is defined as the set of all locally finite $\omega \in (\mathbb{R}^{2d})^I$ such that $\overline{H}(\omega) < +\infty$. The configuration space Ω is equipped with the product topology and with the associated Borel structure. Let us remark that Ω carries a large class of probability measures including all Gibbs random fields with interaction U;

see Refs. 2, 3, and 10. If $\omega_t^{(n)}$ is a sequence of trajectories in Ω then convergence of $\omega_t^{(n)}$ means uniform convergence on compact intervals of time of each of the components $q_k(\omega_t^{(n)})$ and $p_k(\omega_t^{(n)})$. This convergence, however, need not be uniform in $k \in I$. A continuous trajectory $\omega_t : \mathbb{R} \to \Omega$ is called a tempered solution to (1.1) with initial configuration ω if $\omega_0 = \omega$, $\overline{H}(\omega_t)$ is bounded in bounded intervals of time, and the components $q_k(t) = q_k(\omega_t)$, $p_k(t) = p_k(\omega_t)$ are continuously differentiable and satisfy (1.1) for all $k \in I$ and $t \in \mathbb{R}$. In dimensions one and two we have the following.

Theorem 2.1. For each $\omega \in \Omega$ there exists a unique tempered solution $\omega_t = T_t\omega$ with initial configuration ω. This solution can be obtained as the limit of solutions to finite subsystems, and $T_t : \mathbb{R} \times \Omega \to \Omega$ is a group of measurable transformations of Ω onto itself.

The basic *a priori* bound implying Theorem 1.3 for $d \leqslant 2$ can be stated as follows. Let $\Omega_h = [\omega \in \Omega : \overline{H}(\omega) \leqslant h]$; then for each $h > 0$ and $T > 0$ there exists a finite $\bar{h} = \bar{h}(h, T)$ such that $T_t\omega \in \Omega_{\bar{h}}$ if $\omega \in \Omega_h$ and $|t| \leqslant T$. We suspect that $T_t\omega$ is not a continuous function of $\omega \in \Omega$, but a relatively straightforward iteration procedure shows that the restriction of T_t to any nonvoid Ω_h is already a continuous function of $\omega \in \Omega_h$; see Ref. 3.

If $\varphi : \Omega \to \mathbb{R}$ then $T_t\varphi$ denotes the translate of φ by a time t, i.e., $(T_t\varphi)(\omega) = \varphi(T_t\omega)$. Of course, $T_t\omega$ is a jointly measurable function of ω and t; thus T_t maps measurable functions into measurable ones. Further regularity properties of T_t are more sophisticated than those we have for finite systems. For example, Feller continuity and strong continuity of T_t hold only in the following sense. Let $\mathbb{C}_0(\Omega)$ denote the space of $\varphi : \Omega \to \mathbb{R}$ such that the restriction of φ to any nonvoid Ω_h is continuous, then $T_t\mathbb{C}_0(\Omega) = \mathbb{C}_0(\Omega)$ expresses a kind of Feller continuity. It is natural to equip $\mathbb{C}_0(\Omega)$ with a scale $|\varphi|_h = \sup[|\varphi(\omega)| : \omega \in \Omega_h]$ of seminorms, then T_t is strongly continuous with respect to each of these seminorms; cf. Ref. 4.

Conservation of differentiability properties is also an interesting question. Let $D_k\varphi$ denote the vector of partial derivatives of a differentiable $\varphi : \Omega \to \mathbb{R}$ with respect to coordinates of q_k and p_k, i.e., $D_k\varphi : \Omega \to \mathbb{R}^{2d}$, and introduce $\mathbb{C}_1(\Omega)$ as the space of $\varphi \in \mathbb{C}_0(\Omega)$ such that $D_k\varphi$ exists and belongs to $\mathbb{C}_0(\Omega)$ for each $k \in I$, and for each $h > 0$ there exist positive constants K_h and δ_h such that $|(D_k\varphi)(\omega)| \leqslant K_h \exp(-\delta_h |q_k(\omega|)$ for all $\omega \in \Omega_h$. Then we have the following.

Theorem 2.3. If U is twice continuously differentiable on the set $[U < +\infty]$ then $T_t\mathbb{C}_1(\Omega) = \mathbb{C}_1(\Omega)$ for all $t \in \mathbb{R}$.

The evolution of a probability measure P with $P(\Omega) = 1$ is defined as $(PT_t)(E) = P(T_{-t}E)$ for measurable subsets E of Ω. It is well known (see Refs. 6 and 7) that $PT_t = P$ if P is a Gibbs random field with interaction U.

Since $\mathbb{C}_1(\Omega)$ is obviously dense in $\mathbb{L}_2(P)$, Theorem 2.3 implies essential anti-self-adjointness of the Liouville operator; see Refs. 10 and 15.

In the next section the appropriate spatial cutoff of total energy will be described. Then we prove the fundamental *a priori* bound, and investigate dependence of solutions on initial data.

3. CUTOFF OF TOTAL ENERGY

Let $0 < \lambda < 1$ and choose a continuously differentiable nonincreasing function $\varphi : \mathbb{R} \to (0,1)$ such that $\varphi(u) = e^{\lambda(1-u)}$ if $u \geqslant 2$, $\varphi(u) = (1 + \lambda/2)e^{-\lambda}$ if $u \leqslant 1$, and φ is concave for $u \leqslant 2$. Notice that $\varphi(u) \leqslant e^{\lambda(1-u)}$ and $0 < -\varphi'(u) \leqslant \lambda e^{\lambda(1-u)}$. Thus if $\sigma > 0$ then

$$f(x,\sigma) = \int_{\mathbb{R}^d} \varphi(|x-y|/\sigma) e^{-\lambda|y|}\, dy \tag{3.1}$$

is well defined for all $x \in \mathbb{R}^d$. In the proof of the *a priori* bound $f(x-\mu,\sigma)$ will be used as a smooth version of the indicator function of the d-dimensional sphere with center μ and radius σ. From now on we assume that $\sigma \geqslant 2$; then an easy calculation yields

$$f(x-\mu,\sigma) \leqslant c_1 \exp\left[\lambda(1 - |x-\mu|/\sigma)\right] \tag{3.2}$$

for all x, $\mu \in \mathbb{R}^d$ and $\sigma \geqslant 2$, while

$$f(x-\mu,\sigma) \geqslant c_2 > 0 \tag{3.3}$$

if $|x-\mu| \leqslant \sigma$. Here c_1 and c_2 depend only on λ and d, the value of λ will be specified later. The corresponding version of total energy is defined as

$$W(\omega, \mu, \sigma) = \sum_{k \in I} f(q_k - \mu, \sigma) W_k(\omega) \tag{3.4}$$

where $q_k = q_k(\omega)$, $p_k = p_k(\omega)$, and

$$W_k(\omega) = 2A + |p_k|^2 + \sum_{j \neq k} U(q_k - q_j) \tag{3.5}$$

with A the same as in (1.3); the logarithmic fluctuation of W reads as

$$\overline{W}(\omega) = \sup_{\mu} \sup_{\sigma \geqslant 2g(|\mu|)} \sigma^{-d} W(\omega, \mu, \sigma) \tag{3.6}$$

A similar cutoff of additive Liapunov functions was used in Ref. 5.

In the forthcoming calculations the following elementary properties of f and W are needed. Observe first that

$$f(x,\sigma) \leqslant e^{\lambda|x-y|} f(y,\sigma) \tag{3.7}$$

and

$$f'(x,\sigma) \leqslant e^{\lambda|x-y|} f'(y,\sigma) \tag{3.8}$$

where f' denotes the derivative of f with respect to σ. Indeed, as $-|x-z| \leqslant |x-y| - |y-z|$, we obtain that

$$f(x,\sigma) = \int \varphi(|z|/\sigma) e^{-\lambda|x-z|}\,dz$$
$$\leqslant e^{\lambda|x-y|} \int \varphi(|z|/\sigma) e^{-\lambda|y-z|}\,dz = e^{\lambda|x-y|} f(y,\sigma)$$

On the other hand, as

$$f'(x,\sigma) = -\int \varphi'(|x-z|/\sigma) \frac{|x-z|}{\sigma^2} e^{-\lambda|z|}\,dz \tag{3.9}$$

(3.8) follows in the same way. Let grad f denote the gradient of f with respect to $x \in \mathbb{R}^d$, since $\varphi'(u) = 0$ if $u \leqslant 1$, (3.9) implies that

$$|\text{grad}\, f(x,\sigma)| \leqslant f'(x,\sigma) \tag{3.10}$$

We also need that

$$g(|x|)\text{grad}\, f(x-\mu,\sigma)| \leqslant 4g(|\mu| + \sigma) f'(x-\mu,\sigma) \tag{3.11}$$

first we prove

$$|x|\,|\text{grad}\, f(x,\sigma)| \leqslant 4\sigma f'(x,\sigma) \tag{3.12}$$

In view of (3.10) we may assume that $|x| \geqslant 4\sigma$. Let $D_1 = [y \in \mathbb{R}^d : |y| \leqslant |x-y|]$ and $D_2 = \mathbb{R}^d \setminus D_1$, then $|y| \geqslant |x|/2$ if $y \in D_2$, and $-\varphi'(u) \leqslant \lambda e^{\lambda(1-u)}$ implies

$$-\varphi'(|y|/\sigma) e^{-\lambda|x-y|} \leqslant -\varphi'(|x-y|/\sigma) e^{-\lambda|y|}$$

for $y \in D_1$, consequently

$$\sigma^2 f'(x,\sigma) = -\int_{\mathbb{R}^d} \varphi'(|y|/\sigma)|y| e^{-\lambda|x-y|}\,dy$$
$$\geqslant -\frac{|x|}{2} \int_{D_2} \varphi'(|y|/\sigma) e^{-\lambda|x-y|}\,dy$$
$$\geqslant -\frac{|x|}{4} \int_{\mathbb{R}^d} \varphi'(|y|/\sigma) e^{-\lambda|x-y|}\,dy$$
$$\geqslant \frac{\sigma|x|}{4} \left| \int_{\mathbb{R}^d} -\varphi'(|x-y|/\sigma) \frac{x-y}{\sigma|x-y|} e^{-\lambda|y|}\,dy \right|$$

which proves (3.12). To conclude (3.11) observe that $g(u)/u$ is decreasing if $u > 0$, thus

$$|x|\,|\text{grad}\, f(x-\mu,\sigma)| \leqslant |x-\mu|\,|\text{grad}\, f(x-\mu,\sigma)| + |\mu|\,|\text{grad}\, f(x-\mu,\sigma)|$$
$$\leqslant (4\sigma + |\mu|) f'(x-\mu,\sigma)$$

implies (3.11) as $g(4\sigma + |\mu|) \leqslant g(4) + g(\sigma + |\mu|) \leqslant 4g(\sigma + |\mu|)$.

The superstability of U implies the following.

Lemma 3.1. There exists a $0 < \lambda < 1$ such that

$$W(\omega, \mu, \sigma) \geqslant \frac{B}{4} \sum_{k \in I} f(q_k - \mu, \sigma) N_k(\omega)$$

and

$$\frac{\partial}{\partial \sigma} W(\omega, \mu, \sigma) \geqslant \frac{B}{4} \sum_{k \in I} f'(q_k - \mu, \sigma) N_k(\omega)$$

where $N_k(\omega)$ denotes the number of $j \neq k$ such that $|q_k - q_j| \leqslant r$, while $B > 0$ is the same as in (1.3).

Proof. Introduce

$$\Lambda_u = \left[x \in \mathbb{R}^d : u^{(i)} \leqslant x^{(i)} \leqslant u^{(i)} + mr;\ 1 \leqslant i \leqslant d \right]$$

where $x^{(i)}$ and $u^{(i)}$ are the coordinates of $x, u \in \mathbb{R}^d$, and m is a large natural number. Let P denote the set of pairs $[k, j]$ such that $|q_k - q_j| \leqslant r$, and let P_u be the set of $[k, j] \in P$ such that $k, j \in I_u$, where I_u is the set of particles in Λ_u. If ρ_u denotes the minimum of $f_k = f(q_k - \mu, \sigma)$ for $k \in I_u$, and $\lambda m r \sqrt{d} < \epsilon$, i.e., the diameter of Λ_u is less than ϵ/λ, then using (3.7) and $U(x) \geqslant -a/b$ we obtain for each $[k, j] \in P_u$

$$\begin{aligned}(f_k + f_j) U(q_k - q_j) &= 2\rho_u U(q_k - q_j) + (f_k + f_j - 2\rho_u) U(q_k - q_j) \\ &\geqslant 2\rho_u U(q_k - q_j) - \frac{2a}{b} \rho_u (e^{\epsilon} - 1)\end{aligned}$$

thus (1.3) implies that

$$\begin{aligned}&\sum_{[k,j] \in P_u} (f_k + f_j) U(q_k - q_j) \\ &\quad \geqslant -A \sum_{k \in I_u} \rho_u + \left[B - \frac{2a}{b}(e^{\epsilon} - 1) \right] \sum_{[k,j] \in P_u} \rho_u \\ &\quad \geqslant -A e^{\epsilon} \sum_{k \in I_u} f_k + \left[\frac{B}{2} - \frac{a}{b}(e^{\epsilon} - 1) \right] e^{-\epsilon} \sum_{[k,j] \in P_u} (f_k + f_j) \\ &\quad \geqslant -2A \sum_{k \in I_u} f_k + \frac{B}{3} \sum_{[k,j] \in P_u} (f_k + f_j) \qquad (3.13)\end{aligned}$$

at least if $\epsilon > 0$ is small enough. Now let $z \in \Lambda_0 \cap r\mathbb{Z}^d$, where $\mathbb{Z}^d$ is the d-dimensional integer lattice. Then $[\Lambda_u : u \in z + rm\mathbb{Z}^d]$ is a partition of $\mathbb{R}^d$; thus (3.13) implies

$$W(\omega, \mu, \sigma) \geqslant -\frac{a}{b} \sum_{[k,j] \in P \setminus Q_z} (f_k + f_j) + \frac{B}{3} \sum_{[k,j] \in Q_z} (f_k + f_j) \qquad (3.14)$$

where Q_z is the union of all P_u such that $u \in z + rm\mathbb{Z}^d$. Given $[k, j] \in P$, the number of $z \in \Lambda_0 \cap r\mathbb{Z}^d$ such that $[k, j] \in Q_z$ is certainly larger than

$(m-2)^d$; thus the number of z with $[k, j] \notin Q_z$ is less than $m^d - (m-2)^d \leqslant 2dm^{d-1}$. Consequently, summing both sides of (3.14) over $z \in \Lambda_0 \cap r\mathbb{Z}^d$ we have

$$m^d W(\omega, \mu, \sigma) \geqslant \left[\frac{B}{3}(m-2)^d - \frac{2ad}{b} m^{d-1} \right] \sum_{[k,j] \in P} (f_k + f_j)$$

Since m can be so large that

$$\frac{B}{3}\left(1 - \frac{2}{m}\right)^d - \frac{2ad}{bm} > \frac{B}{4}$$

the first statement follows directly for $0 < \lambda < \epsilon / mr\sqrt{d}$; cf. (3.13). Using (3.8) instead of (3.7), we obtain the second inequality in the same way. ■

In the rest of the paper $\lambda > 0$ will be fixed such that Lemma 3.1 holds true. Finally, let us remark that H and W are equivalent in the following sense. There exist positive constants c_3 and c_4 such that

$$c_3 \overline{H}(\omega) \leqslant \overline{W}(\omega) \leqslant c_4 \overline{H}(\omega) \tag{3.15}$$

for all $\omega \in \Omega$. The first part of (3.15) is a direct consequence of Lemma 3.1. Since $f(x - \mu, \sigma) \leqslant \exp[\lambda(1 - |x - \mu|/\sigma)]$, by (1.3) it follows that

$$W(\omega, \mu, \sigma) \leqslant K_1 \sum_{n=1}^{\infty} H(\omega, \mu, n\sigma) e^{-\lambda n} \leqslant K_1 \sigma^d \overline{H}(\omega) \sum_{n=1}^{\infty} n^d e^{-\lambda n}$$

if $\sigma \geqslant 2g(|\mu|)$, which completes the proof of (3.15).

4. THE *A PRIORI* BOUND

Just as in Refs. 2 and 3, our basic tool is the following partial differential inequality that controls energy flow along tempered solutions in dimensions one and two. Temperedness is needed to exclude influence from infinity. Owing to the above properties of the spatial cutoff of total energy, we can prove a local version of the law of energy conservation. Suppose that $d \leqslant 2$.

Proposition 4.1. There exists a universal constant $K > 0$ such that along any tempered solution ω_t to (0, 1) we have

$$\frac{\partial}{\partial t} W(\omega_t, \mu, \sigma) \leqslant Kg(|\mu| + \sigma)\left[\overline{W}(\omega_t)\right]^{1/2} \frac{\partial}{\partial \sigma} W(\omega_t, \mu, \sigma)$$

for all $t \in \mathbb{R}$, $\mu \in \mathbb{R}^d$, and $\sigma \geqslant 2$.

Proof. Typical notations of previous sections are used without any reference; furthermore $\operatorname{grad} f_k = \operatorname{grad} f(q_k - \mu, \sigma)$ and $f_k' = f'(q_k - \mu, \sigma)$.

Differentiating $W(\omega_t, \mu, \sigma)$ with respect to t we obtain

$$\frac{\partial W}{\partial t} = \sum_{k\in I} \langle \text{grad}\, f_k, p_k \rangle W_k + \frac{1}{2} \sum_{k\in I} \sum_{j\neq k} (f_j - f_k) \langle \text{grad}\, U(q_k - q_j), p_k + p_j \rangle \tag{4.1}$$

In view of (1.2),

$$|W_k(\omega_t)| \leqslant W_k(\omega_t) + \frac{a}{b} N_k(\omega_t)$$

and

$$|f_k - f_j|\, |\text{grad}\, U(q_k - q_j)| \leqslant |\text{grad}\, f(z_{kj} - \mu, \sigma)| (a + bU(q_k - q_j))$$

where z_{kj} is a point between q_k and q_j. On the other hand, if $|q_k - q_j| \leqslant r$ then

$$\max\left[|p_k|, |p_k + p_j| \right] \leqslant K_2 g(|q_k|) \left[\overline{W}(\omega_t) \right]^{1/2}$$

follows from Lemma 3.1, thus using (3.11) and (3.8) we obtain

$$\frac{\partial W}{\partial t} \leqslant K_3 g(|\mu| + \sigma) \left[\overline{W}(\omega_t) \right]^{1/2} \sum_k f_k' \left(W_k(\omega_t) + \frac{a}{b} N_k(\omega_t) \right)$$

whence the statement follows directly by Lemma 3.1. ■

This partial differential inequality can be solved by the method of characteristics in the same way as in Refs. 2 and 3. For reader's convenience we reproduce the main steps. Let $T > 0$, $\sigma \geqslant 2g(|\mu|)$ and define $\rho(t)$ for $0 \leqslant t \leqslant T$ as the unique solution to the integral equation

$$\rho(t) = \sigma + K \int_t^T g(|\mu| + \rho(s)) \left[\overline{W}(\omega_s) \right]^{1/2} ds \tag{4.2}$$

Proposition 4.1 implies that $W(\omega_t, \mu, \rho(t))$ is decreasing, i.e.,

$$W(\omega_T, \mu, \sigma) \leqslant W(\omega_0, \mu, \rho(0)) \tag{4.3}$$

Let $Z(t) = \int_0^t (\overline{W}(\omega_s))^{1/2} ds$; then (4.2) implies

$$\rho(0) \leqslant \sigma + K_4 \left[\sigma + g(\rho(0)) \right] Z(T)$$

whence

$$\rho(0) \leqslant \sigma K_5 \left[1 + g(Z(T)) Z(T) \right]$$

follows by an easy calculation (see Ref. 3), consequently (4.3) turns into

$$\frac{dZ}{dt} \leqslant K_6 \left[\overline{W}(\omega_0) \right]^{1/2} \left[1 + Zg(Z) \right]^{d/2} \tag{4.4}$$

Since

$$\int_0^\infty \left[1 + Zg(Z) \right]^{-d/2} dZ = +\infty$$

as $d \leqslant 2$, (4.4) has a maximal solution which is bounded in finite intervals of time; thus (4.4) yields a similar bound for $\overline{W}(\omega_t)$, too. Exploiting reflection symmetry of (1.1) and of $\overline{W}$, the same bound follows for negative values of time. By means of (3.15) we can formulate the result in terms of H. We have the following.

Proposition 4.2. For each $h > 0$ and $T > 0$ there exists a finite $\bar{h} = \bar{h}(h, T)$ such that $\overline{H}(\omega_0) \leqslant h$ implies $\overline{H}(\omega_t) \leqslant \bar{h}$ for all $|t| \leqslant T$ provided that ω_t is a tempered solution to (1.1).

We need a similar bound for the localization of particles.

Proposition 4.3. If ω_t is a tempered solution to (1.1) and $\overline{H}(\omega_0) \leqslant h$; then $|q_k(\omega_t)| \leqslant (1 + |q_k(\omega_0)|)\exp[T(2\bar{h})^{1/2}]$ for all $k \in I$ and $|t| \leqslant T$; here $\bar{h}$ is the same as in Proposition 4.2.

Proof. Proposition 4.2 yields

$$\left| \frac{dq_k}{dt} \right| \leqslant g(|q_k|)(2\bar{h})^{1/2} \leqslant (1 + |q_k|)(2\bar{h})^{1/2}$$

for $|t| \leqslant T$, which proves the statement. ■

Remark 4.4. Proposition 4.1 can be proven in all dimensions, it is enough to replace $g(|\mu| + \sigma)$ on the right-hand side by $[d/2 + \log(1 + |\mu| + \sigma)]^{d/2}$. However, (4.4) has a global maximal solution only if $d \leqslant 2$. In the d-dimensional case

$$\frac{\partial}{\partial t} W(\omega_t, \mu, \sigma) \leqslant Kg(|\mu| + \sigma)\left[\overline{W}(\omega_t)\right]^{1/d} \frac{\partial}{\partial \sigma} W(\omega_t, \mu, \sigma)$$

would be needed for an *a priori* bound.

Remark 4.5. If $d = 1$ then (1.2) can be replaced by $U \geqslant -a/b$ and

$$|x|\,|U'(x)| \leqslant \left[a + bU(x)\right]^{3/2}$$

Indeed, estimating $|p_k + p_j|[a + bU(q_k - q_j)]^{1/2}$ by a multiple of $g(|q_k|)$ $\overline{W}(\omega_t)$,

$$\frac{\partial W(\omega_t, \mu, \sigma)}{\partial t} \leqslant Kg(|\mu| + \sigma)\overline{W}(\omega_t)\frac{\partial W(\omega_t, \mu, \sigma)}{\partial \sigma}$$

follows in the same way as Proposition 4.1 has been obtained. In the one-dimensional case this differential inequality also implies Proposition 4.2; see Ref. 2. Very singular potentials of this kind are also discussed in Ref. 12, where conditions are stronger.

Remark 4.6. If $d = 1$ and (1.2) is supposed then (4.4) yields a polynomial bound, namely, in Proposition 4.2 we have $\bar{h}(h, T) = K_\epsilon(h + (Th)^{2+\epsilon})$ for any $\epsilon > 0$.

Of course, we are not given tempered solutions in general, but any solution is a tempered one if I is a finite collection of particles. Thus we have uniform bounds which do not depend on the size of the system that means compactness of a suitably chosen sequence of approximate solutions. Hence existence of tempered solutions to (1.1) follows by continuity in a similar way as the Peano theorem is proven. The uniqueness of the tempered solution is obtained by means of the contraction principle; see Ref. 3. In the next section we are going to investigate (1.1) and its first variational system simultaneously.

5. COMPACTNESS AND CONTRACTION PRINCIPLES

The skeleton of some additional arguments can be summarized in the following iteration procedure. Consider a sequence $\delta(t,m)$, $m = 0, 1, \ldots$ of nonnegative and continuous functions on $[0, T]$ such that

$$\delta(t,m) \leqslant \delta(0,m) + L_T g^\sigma(\rho + m) \int_0^t \delta(s, m+1)\, ds \tag{5.1}$$

holds for all $t \leqslant T$ and m with some $\rho > 0$, $\sigma > 0$, and $L_T > 0$. If

$$\delta(t,m) \leqslant Q_T \exp(mQ_T) \tag{5.2}$$

for $t \leqslant T$ and for all $m \geqslant 0$ then (5.1) can be iterated infinitely many times, and we obtain

$$\delta(t,0) \leqslant \sum_{m=0}^{\infty} \delta(0,m) \frac{(L_T t)^m}{m!} g^{\sigma m}(\rho + m) \tag{5.3}$$

for all $t \leqslant T$.

Consider now the first variational system associated to (0.1). If Λ denotes the matrix of second partial derivatives of U then we have

$$\frac{dv_k}{dt} = -\sum_{i \neq k} \Lambda(q_k(\omega_t) - q_j(\omega_t))(u_k - u_j), \qquad \frac{du_k}{dt} = v_k \tag{5.4}$$

for $k \in I$, where $u_k \in \mathbb{R}^d$, $v_k \in \mathbb{R}^d$, and ω_t is a tempered solution to (1.1). Let us remark that (5.4) can be obtained by a formal differentiation of (1.1) with respect to a parameter. This parameter will be chosen as a coordinate of the initial configuration $\omega_0 \in \Omega$; thus $|u_k(0)| \leqslant 1$ and $|v_k(0)| \leqslant 1$ may be supposed for all $k \in I$; but this restriction is not really essential. From now on we consider a fixed initial configuration $\omega_0 \in \Omega$, thus all constants—apart from σ and ρ—will depend on $\overline{H}(\omega_0)$ via Propositions 4.2 and 4.3. First we prove the existence of solutions to the joint system [(1.1), (5.4)] satisfying a reasonable *a priori* bound.

Let $f: \mathbb{R} \to [0, 1]$ be a continuously differentiable nonincreasing function such that $f(u) = 1$ if $u \leqslant r$, $f(u) = 0$ if $u \geqslant 2r$ and $-f'(u) \leqslant f(u - r)$

for all $u \in \mathbb{R}$. It will be convenient to use the $\mathbb{L}_1$-type norm $|x|_1 = |x^{(1)}| + |x^{(2)}| + \cdots + |x^{(d)}|$ for elements $x = (x^{(1)}, x^{(2)}, \ldots, x^{(d)})$ of $\mathbb{R}^d$. Suppose that I is finite and put

$$\delta_k(t,m) = \sum_{j \in I} f(|q_k - q_j| - 2rm)(|u_j|_1 + |v_j|_1) \tag{5.5}$$

where $(q_k, p_k, u_k, v_k)_{k \in I}$ denotes the corresponding solution to (1.1), (5.4). It is obvious that δ_k is an absolutely continuous function of time, thus (5.5) can be differentiated almost everywhere in $[0, T]$. Elements of Λ are bounded in view of (1.4), thus an easy calculation shows that δ_k satisfies (5.1) with $\sigma = 2c + 1$ and $\rho = |q_k(0)|$ for each $k \in I$. The necessary bounds follow from Propositions 4.2 and 4.3; cf. the similar proofs in Refs. 2 and 3. As a consequence, we obtain that

$$\delta_k(t, 0) \leqslant \sum_{m=0}^{\infty} \delta_k(0, m) \frac{(L_T t)^m}{m!} g^{\sigma m}(|q_k(0)| + m) \tag{5.6}$$

Since $\delta_k(0, m)$ is bounded by a multiple of $g^d(|q_k(0)|)(1 + m)^d$ in view of (1.3), we have obtained an effective *a priori* bound for (5.4). In fact, it would be possible to study (5.4) in the space of sequences $(q_k, p_k, u_k, v_k)_{k \in I}$ such that $w = (q_k, p_k)_{k \in I}$ belongs to Ω, and

$$\sup_{k \in I} (1 + |q_k|)^{-1} \log(|u_k| + |v_k|) < +\infty \tag{5.7}$$

Just as in the case of usual configurations, the notations $w = (u_k, v_k)_{k \in I}$ and $u_k = u_k(w)$, $v_k = v_k(w)$, $u_k(t) = u_k(w_t)$, $v_k(t) = v_k(w_t)$ will be used if $w, w_t \in (\mathbb{R}^{2d})^I$ are associated to (5.4).

Theorem 5.1. Consider an infinite collection I of particles in dimensions one or two; let $\omega_0 \in \Omega$, $w_0 \in (\mathbb{R}^{2d})^I$ and suppose that $|u_j(w_0)|_1 + |v_j(w_0)|_1 = 1$ for a given $j \in I$, while $|u_k(w_0)|_1 + |v_k(w_0)|_1 = 0$ if $k \neq j$. Then there exists at least one solution (ω_t, w_t) to (1.1), (5.4) with initial condition (ω_0, w_0) such that ω_t is a tempered solution to (1.1), and w_t satisfies

$$|u_k(w_t)|_1 + |v_k(w_t)|_1 \leqslant \sum_{m=d_{kj}}^{\infty} \frac{(L_T t)^m}{m!} g^{\sigma m}(|q_k(\omega_0)| + m)$$

for all $k \in I$ and $|t| \leqslant T$ with $\sigma = 2c + 1$ and L_T depending only on $\overline{H}(\omega_0)$ and T. Here d_{kj} denotes the integer part of $|q_k(\omega_0) - q_j(\omega_0)|/2r$.

Proof. Let $I_n = I_n(\omega_0)$ be obtained from I by deleting the particles k such that $|q_k(\omega_0)| > nr$, $n = 1, 2, \ldots,$ and let $(\omega_t^{(n)}, w_t^{(n)})$ denote the corresponding solution to (1.1), (5.4). Then $\delta_k(0, m) = 0$ if $m < d_{kj}$ and $\delta_k(0, m) \leqslant 1$ otherwise, thus (5.6) implies the *a priori* bound of Theorem 5.1 for each $w_t^{(n)}$; for the sequence $\omega_t^{(n)}$ Propositions 4.2 and 4.3 can be applied.

Therefore the Arzela–Ascoli theorem implies the relative compactness of the sequence $(\omega_t^{(n)}, w_t^{(n)})$ in the uniform product topology of trajectories. This means that we can select a subsequence n_i such that each component $q_k(\omega_t^{(n_i)})$, $p_k(\omega_t^{(n_i)})$, $u_k(w_t^{(n_i)})$ and $v_k(w_t^{(n_i)})$ converges uniformly on $[-T, T]$ as $i \to +\infty$. Let (ω_t, w_t) denote the limit, since T can be arbitrary, (ω_t, w_t) can be defined for all $t \in \mathbb{R}$. Since $\overline{H}$ is lower semicontinuous, and $|u_k|_1 + |v_k|_1$ is continuous, the *a priori* bounds remain in force for (ω_t, w_t), too, thus exploiting the continuity of the right-hand side of the integral version of (1.1), (5.4) we obtain that (ω_t, w_t) really satisfies (1.1), (5.4). ■

The next step is to prove the uniqueness of tempered solutions to (1.1). Let ω_t and $\overline{\omega}_t$ be two tempered solutions with $\omega_0, \overline{\omega}_0 \in \Omega_h$, and introduce

$$d_R(\omega_t, \overline{\omega}_t, m) = \sum_{k \in I} f(|q_k| - R - 2rm) f(|\overline{q}_k| - R - 2rm) \times \left[|q_k - \overline{q}_k|_1 + |p_k - \overline{p}_k|_1 \right] \tag{5.8}$$

where $R > 0$ and $\overline{q}_k = q_k(\overline{\omega}_t)$, $\overline{p}_k = p_k(\overline{\omega}_t)$. It is easy to check (cf. Refs. 2 and 3) that $\delta(t, m) = d_R(\omega_t, \overline{\omega}_t, m)$ satisfies (5.1) and (5.2) with $\rho = 0$ and $\sigma = 2c + 1$; therefore (5.3) yields

$$d_R(\omega_t, \overline{\omega}_t, 0) \leqslant \sum_{m=0}^{\infty} d_R(\omega_0, \overline{\omega}_0, m) \frac{(L_T t)^m}{m!} g^{\sigma m}(m) \tag{5.9}$$

for all $|t| \leqslant T$ and $R > 0$; L_T depends on h, R, and T. Since R can be as large as necessary, and Proposition 4.3 controls the displacement of particles, (5.9) implies the uniqueness of tempered solutions. However, this is possible only if the sequence $\omega_t^{(n)}$ defined in the proof of Theorem 5.1 has only one limit point, i.e., $\omega_t^{(n)}$ converges to the unique tempered solution, which completes the proof of Theorem 2.1.

Since

$$d_R(\omega_0, \overline{\omega}_0, m) \leqslant K(1 + h)(R + m)^{d+1} \tag{5.10}$$

with a universal constant K, we also have continuity of $T_t\omega$ as a function of $\omega \in \Omega_h$ for each $h > 0$, whence $T_t C_0(\Omega) = C_0(\Omega)$ follows directly for all $t \in \mathbb{R}$.

The uniqueness of such solutions to (1.1), (5.4) which satisfy the conclusions of Theorem 5.7 can be proven in a quite similar way, but we do not need this result. Now we are going to show that tempered solutions are differentiable functions of the initial configuration, and the derivatives satisfy (5.4).

Consider a given initial configuration $\omega \in \Omega_h$ and let $D_j^{(i)}$, $i = 1, 2, \ldots, 2d$, $j \in I$ denote differentiation with respect to the i coordinate of $(q_j(\omega), p_j(\omega)) \in \mathbb{R}^{2d}$. First we show that

$$u_k(t) = D_j^{(i)} q_k(T_t\omega), \qquad v_k(t) = D_j^{(i)} p_k(T_t\omega) \tag{5.11}$$

exist and satisfy (5.4) and the *a priori* bound of Theorem 5.1 for all i and j. This is certainly true for the approximate solutions $w_t^{(n)}$ defined in the proof of Theorem 5.1. Writing (5.11) in an integral form we see that (5.11) extends also to limit points of $(\omega_t^{(n)}, w_t^{(n)})$, which proves that u_k and v_k are correctly defined by (5.11) and satisfy the *a priori* bound of Theorem 5.1. Let $\varphi \in \mathbb{C}_1(\Omega)$ and $|t| \leqslant T$, then from the chain rule we obtain that

$$|D_j^{(i)}\varphi(T_t\omega)| \leqslant \sum_{k\in I} |(D_k\varphi)(T_t\omega)|_1(|u_k(t)|_1 + |v_k(t)|_1)$$

where

$$|(D_k\varphi)(T_t\omega)|_1 \leqslant C_{\bar h}\exp(-\delta_{\bar h}|q_k(T_t\omega)|) \leqslant C\exp(-\delta|q_k(\omega)|)$$

in view of Propositions 4.2 and 4.3 with some new $C < \infty$ and $\delta > 0$ depending on h and T. On the other hand, Theorem 5.1 applies to $|u_k(t)|_1 + |v_k(t)|_1$ as $|t| \leqslant T$; thus

$$|D_j^i\varphi(T_t\omega)| \leqslant C\sum_{k\in I} e^{-\delta|q_k|} \sum_{m=d_{kj}}^{\infty} \frac{(L_T T)^m}{m!} g^{\sigma m}(|q_k| + m)$$

Since $-|q_k| \leqslant |q_k - q_j| - |q_j|$, $|q_k| \leqslant |q_k - q_j| + |q_j|$, and $|q_k - q_j| \leqslant 2r(m+1)$, rearranging the above sum we obtain from (1.3) that

$$\begin{aligned}|D_j^{(i)}\varphi(T_t\omega)| &\leqslant C_1 e^{-\delta|q_j|} \sum_{m=0}^{\infty} \frac{(m+1)^d}{m!} (Mg^{\sigma}(|q_j| + m))^m \\ &\leqslant C_1 e^{-\delta|q_j|} \sum_{m=0}^{\infty} \frac{1}{m!}\left[M_1 g^{\sigma}(|q_j| + m)\right]^m \qquad (5.12)\end{aligned}$$

with some C_1 and M_1 depending on h and T. Finally, the Hölder inequality implies

$$g^{\sigma m}(\rho + m) \leqslant (g(\rho) + g(m))^{\sigma m} \leqslant 2^{\sigma m}(g^{\sigma m}(\rho) + g^{\sigma m}(m))$$

thus (5.12) results

$$\begin{aligned}|D_j^{(i)}\varphi(T_t\omega)| &\leqslant C_1 e^{-\delta|q_j|} \sum_{m=0}^{\infty} \frac{1}{m!}(2^{\sigma}M_1 g^{\sigma}(|q_j|))^m \\ &\quad + C_1 e^{-\delta|q_j|} \sum_{m=0}^{\infty} \frac{1}{m!}(2^{\sigma}M_1 g^{\sigma}(m))^m \\ &= C_1\exp\left[M_2 g^{\sigma}(|q_j|) - \delta|q_j|\right] + C_2 e^{-\delta|q_j|}\end{aligned}$$

Since $\rho^{-1}g^{\sigma}(\rho)$ goes to zero as $\rho \to +\infty$, we have a C_3 such that

$$|D_j^{(i)}\varphi(T_t\omega)| \leqslant C_3\exp\left[-\frac{\delta}{2}|q_j(\omega)|\right] \qquad (5.13)$$

holds for all $|t| \leqslant T$, $\bar H(\omega) \leqslant h$, $i = 1, 2, \ldots, 2d$ and $j \in I$; C_3 and δ depend only on h and T. Since T and h are arbitrary, (5.13) implies Theorem 2.3 directly.

6. ANHARMONIC SYSTEMS

In this section we are going to explain how our method works in the case of lattice models with attractive interactions; see Refs. 9 and 11–13. A continuous spin system with $I = \mathbb{Z}^d$ is considered in an external field φ, neighboring spins interact by a symmetric pair potential U. Then the equations of motion are

$$\begin{aligned} \frac{dp_k}{dt} &= -\varphi'(q_k) - \sum_{j:|j-k|=1} U'(q_k - q_j) \\ \frac{dq_k}{dt} &= p_k, \qquad k \in \mathbb{Z}^d \end{aligned} \tag{6.1}$$

where $q_k, p_k \in \mathbb{R}$, $\varphi : \mathbb{R} \to [0, +\infty)$ and $U : \mathbb{R} \to [0, +\infty)$ are continuously differentiable, $U(x) = U(-x)$ and we assume that

$$|U'(x)| \leqslant a(1 + U(x))^b \tag{6.2}$$

with $b \leqslant 1/2 + 1/d$, furthermore a local Lipschitz condition

$$\begin{aligned} &|\varphi'(x) - \varphi'(y)| + |U'(x) - U'(y)| \\ &\quad \leqslant L\left[1 + \varphi(x) + \varphi(y) + U(x) + U(y)\right]^c |x - y| \end{aligned} \tag{6.3}$$

holds with some $c > 0$, $L > 0$.

Configurations of the system are represented as $\omega = (q_k, p_k)_{k \in \mathbb{Z}^d}$, and

$$W(\omega, \mu, \sigma) = \sum_k f(k - \mu, \sigma) W_k(\omega) \tag{6.4}$$

where

$$W_k(\omega) = 1 + p_k^2 + 2\varphi(q_k) + \sum_{j:|j-k|=1} U(q_k - q_j) \tag{6.5}$$

and f is the same as in (3.4); the value of $0 < \lambda < 1$ is not important here. Now let

$$\overline{W}(\omega) = \sup_{\mu} \sup_{\sigma \geqslant 2g(|\mu|)} \sigma^{-d} W(\omega, \mu, \sigma) \tag{6.6}$$

and define Ω as the set of configurations ω such that $\overline{W}(\omega) < +\infty$.

Theorem 6.1. For each $\omega \in \Omega$ there exists a solution $\omega_t = T_t\omega$ to (6.1) with initial condition $\omega_0 = \omega$. This solution can be obtained as the limit of solutions to finite subsystems of (6.1), and there is no other solution $\bar{\omega}_t$ such that $\bar{\omega}_0 = \omega$, and $\overline{W}(\bar{\omega}_t)$ remains bounded in finite intervals of time. Moreover, there exists a continuous $\bar{w} : [0, \infty) \times \mathbb{R} \to [0, \infty)$ such that $\overline{W}(T_t\omega) \leqslant \bar{w}(\overline{W}(\omega), t)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$.

Proof. If ω_t is a solution then

$$\frac{\partial W}{\partial t} = \frac{1}{2} \sum_{k} \sum_{|j-k|=1} (f_j - f_k) U'(q_k - q_j)(p_k + p_j) \tag{6.7}$$

and (6.2) implies

$$|U'(q_k - q_j)| \, |p_k + p_j| \leqslant a(1 + U(q_k - q_j))^b |p_k + p_j|$$
$$\leqslant \frac{a}{2} (1 + U(q_k - q_j))^{1/d} (2 + U(q_k - q_j) + p_k^2 + p_j^2)$$

thus using (3.8) and (3.11) in the same way as in the proof of Proposition 4.1 we obtain

$$\frac{\partial}{\partial t} W(\omega_t, \mu, \sigma) \leqslant Kg(|\mu| + \sigma) \left[\overline{W}(\omega_t) \right]^{1/d} \frac{\partial}{\partial \sigma} W(\omega_t, \mu, \sigma) \tag{6.8}$$

Repeating the proof of Proposition 4.2 we see that (6.8) results in an *a priori* bound in all dimensions; thus a simplified version of the proof of Theorem 2.1 yields the statements of Theorem 6.1. ■

Theorem 6.1 improves the results of Ref. 13, where $b < 1/2 + 1/d$ was supposed, and the *a priori* bound is less effective. Theorem 2.2 can also be extended to lattice systems.

Lattice models with repulsive singular interactions can be treated by means of methods developed for point systems in Refs. 2 and 3; the existence of global solutions can be proven in this way again only in dimensions one and two.

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